

IDENTITIES INVOLVING WEIGHTED CATALAN, SCHRÖDER AND MOTZKIN PATHS

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ABSTRACT. In this paper, we investigate the weighted Catalan, Motzkin and Schröder numbers together with the corresponding weighted paths. The relation between these numbers is illustrated by three equations, which also lead to some known and new interesting identities. To show these three equations, we provide combinatorial proofs. One byproduct is to find a bijection between two sets of Catalan paths: one consisting of those with k valleys, and the other consisting of k \mathbf{N} steps in even positions.

1. INTRODUCTION

The *Catalan numbers*

$$C_n := \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 0,$$

play a very important role in combinatorics. In [8, Exercise 6.19], Stanley listed 66 kinds of different combinatorial interpretations of C_n . For example, C_n counts all Catalan paths of order n , which are the plane lattice paths from $(0, 0)$ to (n, n) which consisting of up steps $\mathbf{N} := (0, 1)$ and horizontal steps $\mathbf{E} := (1, 0)$ and never going below the line $y = x$. A classical result on the Catalan numbers is the Touchard's identity:

$$(1.1) \quad C_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k 2^{n-1-2k},$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

Intimately related to Catalan numbers are *Narayana numbers*, which are defined by

$$N_{n,k} := \frac{1}{n} \binom{n}{k} \binom{n}{k+1}, \quad 0 \leq k \leq n-1.$$

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In combinatorics, $N_{n,k}$ counts all Catalan paths of order n having exactly k valleys ([7, Section 2.4.2]), and thus we have

$$\begin{aligned} C_n &= N(n, 0) + N(n, 1) + \cdots + N(n, k-1) \\ &= \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1}. \end{aligned}$$

In [2], Coker derived the following two identities using generating functions:

$$(1.2) \quad \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^k = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k (1+x)^{n-1-2k},$$

$$(1.3) \quad \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^{2k} (1+x)^{2(n-1-k)} = \sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1} x^k (1+x)^k.$$

It is easy to see that the Touchard's identity (1.1) follows from (1.2) in the case when $x = 1$.

Coker [2] proposed the problems of finding combinatorial interpretations of these two identities. In [4], Chen, Yan and Yang answered Coker's problems and gave the bijective proofs of Coker's identities. The key ingredient of their proofs is the weighted Motzkin paths. A *Motzkin path* of order n is a lattice path from $(0, 0)$ to (n, n) , which never passes below the line $y = x$ and consists of double up steps $\mathbf{N}_2 = (0, 2)$, double horizontal steps $\mathbf{E}_2 := (2, 0)$ and diagonal steps $\mathbf{D} := (1, 1)$. The number of all Motzkin paths of order n is counted by the *Motzkin number*

$$M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k, \quad n \geq 0.$$

The Motzkin numbers also have some nice combinatorial properties. For example, in [6], Donaghey and Shapiro listed 14 kinds of different combinatorial interpretations of M_n .

A *weighted lattice path* of order n is an n -ordered lattice path where some of the steps are assigned with weights. For example, we can obtain an (a, b) -*Motzkin path* by assigning weight a to each \mathbf{D} step and weight b to each \mathbf{E}_2 step on a Motzkin path. Denote the set of all Motzkin paths and the set of all (a, b) -Motzkin paths of order n by $\text{Mot}(n)$ and $\text{Mot}_n(a, b)$ respectively. The weight of a path P , denoted by $\text{wt}(P)$, is the product of all the weights assigned on its steps. The weight of a set of paths is the sum of the weights of all the paths in the set. Define the (a, b) -*Motzkin number* $M_n^{(a,b)}$ to be the weight of the set $\text{Mot}_n(a, b)$ and then we have

$$\begin{aligned} M_n^{(a,b)} &:= \sum_{P \in \text{Mot}_n(a,b)} \text{wt}(P) = \sum_{P \in \text{Mot}_n(a,b)} a^{\text{diag}(P)} b^{\text{ea}_2(P)} \\ (1.4) \quad &= \sum_{k=0}^{\lfloor n/2 \rfloor} C_k \binom{n}{2k} a^{n-2k} b^k, \end{aligned}$$

where statistics diag and ea_2 respectively count the number of diagonal steps and double horizontal steps in the Motzkin path P . To get equation (1.4), consider the weight of the subset of $\text{Mot}_n(a, b)$ consisting of paths with exactly k \mathbf{N}_2 's steps, which equals $C_k \binom{n}{2k} a^{n-2k} b^k$ since

- (i) there are $(n - 2k)$ diagonal steps for each path in the subset and each \mathbf{D} has weight a ,
- (ii) there are $\binom{n}{n-2k}$ ways to insert diagonal steps in a Catalan path of order $2k$ to form a path in $\text{Mot}(n)$,
- (iii) there are C_k such Catalan paths of order $2k$ with double up steps \mathbf{N}_2 and double horizontal steps \mathbf{E}_2 with weight b on each one.

Clearly $M_n = M_n^{(1,1)}$. Substituting $a = 0$, $b = 1$ in (1.4), we have

$$(1.5) \quad M_n^{(0,1)} = \begin{cases} C_{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, we also get an equivalent form of the Touchard identity by specializing $a = 2$, $b = 1$:

$$(1.6) \quad M_n^{(2,1)} = C_{n+1}.$$

Motivated by (1.5) and (1.6), it is natural to ask whether there exists a common generalization of (1.2) and (1.3). Here we give such an identity in the following theorem using (a, b) -Motzkin numbers, which will be proved in section 2.

Theorem 1.1. *Suppose that $n \geq 1$ and $a, b \in \mathbb{C}$. Then*

$$(1.7) \quad \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^{2k} b^{n-1-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} M_k^{(a,b)} x^k (x^2 - ax + b)^{n-1-k}.$$

Clearly (1.2) follows from (1.7) in the case when $a = 0$, $b = 1$. Noticing that $M_k^{(2(1+x), (1+x)^2)} = M_k^{(2,1)} (1+x)^k$ and by specializing $a = 2(1+x)$, $b = (1+x)^2$, we get (1.3).

Another important combinatorial sequence arising from the lattice path enumeration consists of the *Schröder numbers* S_n , which counts the number of all Schröder paths of order n . An n -ordered *Schröder path* is a lattice path from $(0, 0)$ to (n, n) , which never passes below the line $y = x$ and consist of up steps $\mathbf{N} = (0, 1)$, horizontal steps $\mathbf{E} := (1, 0)$ and diagonal steps $\mathbf{D} := (1, 1)$. We mention that $\binom{n+k}{k} C_k$ counts all Schröder paths of order n having exactly k \mathbf{E} steps, since

- (i) a Schröder paths of order n has exactly k \mathbf{E} steps if and only if it contains exactly $(n + k)$ steps,
- (ii) there are $\binom{n+k}{n-k}$ ways to choose $(n - k)$ \mathbf{D} steps,
- (iii) removing all diagonal steps we get a Catalan path of order k .

Therefore we have

$$S_n = \sum_{k=0}^n \binom{n+k}{k} C_k, \quad n \geq 0.$$

Furthermore, we also have the identity $S_n = 2M_{n-1}^{(3,2)}$ [3, 10]. Motivated by (1.7), we obtain the following theorem.

Theorem 1.2. *Suppose that $n \geq 1$ and $a, b \in \mathbb{C}$. Then*

$$(1.8) \quad \sum_{k=0}^n \binom{n+k}{2k} C_k x^{2k} (b - x^2)^{n-k} = b \sum_{k=0}^{n-1} \binom{n-1}{k} M_k^{(a,b)} x^k (x^2 - ax + b)^{n-1-k}.$$

Setting $a = 0$, $b = 1$ and replacing x^2 by x , we get a special case of (1.8) as the following identity:

$$(1.9) \quad \sum_{k=0}^n \binom{n+k}{2k} C_k x^k (1-x)^{n-k} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k x^k (1+x)^{n-1-2k}.$$

Similarly, letting $a = 2(1+x)$ and $b = (1+x)^2$, we can get

$$(1.10) \quad \sum_{k=0}^n \binom{n+k}{2k} C_k x^{2k} (1+2x)^{n-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1} x^k (1+x)^{k+2}.$$

In particular, setting $x = \frac{1}{2}$ in (1.9), we have

$$(1.11) \quad S_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k 2^{k+1} 3^{n-1-2k},$$

which is exactly the identity $S_n = 2M_{n-1}^{(3,2)}$.

In order to show Theorem 1.1 and 1.2, we need to use the weighted Catalan numbers and the weighted Schröder numbers. An (a, b) -Schröder path of order n is a Schröder path P of order n , whose **D** steps are assigned with weight a and **E** steps are assigned with weight b . We denote by $\text{Sch}_n(a, b)$ the set of all (a, b) -Schröder paths of order n . Define the n^{th} (a, b) -Schröder number $S_n^{(a,b)}$ to be the weight of the set $\text{Sch}_n(a, b)$ as follows

$$S_n^{(a,b)} := \sum_{P \in \text{Sch}_n(a,b)} \text{wt}(P).$$

Let $\text{ea}(P)$ denote the numbers of **E** steps of P . Since the number of n -th Schröder paths having k diagonal steps is given by $\frac{1}{n} \binom{n}{k} \binom{2n-k}{n-1}$ for $0 \leq k \leq n$ ([1]), we have

$$(1.12) \quad \begin{aligned} S_n^{(a,b)} &= \sum_{P \in \text{Sch}_n(a,b)} a^{\text{diag}(P)} b^{\text{ea}(P)} \\ &= \sum_{k=0}^n \binom{n+k}{2k} C_k a^{n-k} b^k. \end{aligned}$$

Assume that P is a Catalan path of order n . We may write $P = p_1 p_2 \cdots p_{2n}$, where $p_1, \dots, p_{2n} \in \{\mathbf{N}, \mathbf{E}\}$. If $p_i = \mathbf{E}$ and $p_{i+1} = \mathbf{N}$, then $p_i p_{i+1}$ forms a valley of P . By assigning each **E** step of P which is followed immediately by an **N** step with weight b , and each other **E** step with weight a , we get the so called *valley type* (a, b) -Catalan path. The set of all valley type (a, b) -Catalan paths of order n is denoted by

$\text{Cat}_n(a, b)$. The corresponding n^{th} valley type (a, b) -Catalan number $C_n^{(a,b)}$ is defined to be the weight of $\text{Cat}_n(a, b)$ as follows

$$C_n^{(a,b)} := \sum_{P \in \text{Cat}_n(a,b)} \text{wt}(P).$$

Since the number of n -ordered Catalan paths having exactly k valleys coincides with the Narayana number $N(n, k)$, we get

$$\begin{aligned} C_n^{(a,b)} &= \sum_{P \in \text{Cat}_n(a,b)} a^{n-\text{valley}(P)} b^{\text{valley}(P)} \\ (1.13) \quad &= \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} a^{n-k} b^k, \end{aligned}$$

where

$$\text{valley}(P) = \#\{1 \leq i \leq n-1 : p_i = \mathbf{E}, p_{i+1} = \mathbf{N}\}.$$

The main result in this paper is to establish the identities between $C_n^{(a,b)}$, $M_n^{(a,b)}$ and $S_n^{(a,b)}$, which are given in the following theorem.

Theorem 1.3. *Suppose that $a, b \in \mathbb{C}$, then we have*

$$(1.14) \quad C_n^{(a,b)} = a M_{n-1}^{(a+b, ab)},$$

$$(1.15) \quad b S_n^{(a,b)} = (a+b) C_n^{(b, a+b)},$$

$$(1.16) \quad S_n^{(a,b)} = (a+b) M_{n-1}^{(a+2b, ab+b^2)}.$$

It is worth mentioned that Theorem 1.3 not only generalizes the known facts that $C_n = M_{n-1}^{(2,1)}$ and $S_n = 2M_{n-1}^{(3,2)}$, but also can be used to show Theorem 1.1 and 1.2. Furthermore, as we shall see later in Section 3, more explicit expressions involving C_n , S_n and M_n can be deduced from Theorem 1.3.

This paper is organized as follows. In Section 2, we prove how Theorem 1.3 implies Theorems 1.1 and 1.2. In Section 3, we give some applications of Theorems 1.1-1.3, which lead to some known and new identities. Sections 4 and 5 are the main part of this paper. In these two sections we are going to give combinatorial proofs for (1.14) and (1.15) in Theorem 1.3, which lead to (1.16). We first give a bijection between the set of Catalan paths with k valleys and that with k \mathbf{N} steps in even positions in Section 4. This allows us to give another combinatorial interpretation of $C_n^{(a,b)}$. In Section 5, we give the combinatorial proof of Theorem 1.3.

2. PROOFS OF THEOREM 1.1 AND 1.2 USING THEOREM 1.3

In this section, we prove Theorems 1.1 and 1.2 under the assumption that Theorem 1.3 holds. To do that, we need the following lemma.

Lemma 2.1. *For $n \geq 0$ and $a_1, a_2, b \in \mathbb{C}$,*

$$(2.1) \quad M_n^{(a_1+a_2, b)} = \sum_{k=0}^n \binom{n}{k} M_k^{(a_1, b)} a_2^{n-k}.$$

Proof. Let $\text{Mot}_n(\{a_1, a_2\}, b)$ denote the set of all Motzkin paths of order n , where each \mathbf{E}_2 step is assigned with weight b and each \mathbf{D} step is assigned with weight either a_1 or a_2 . The paths in $\text{Mot}_n(\{a_1, a_2\}, b)$ are called $(\{a_1, a_2\}, b)$ -Motzkin paths.

For an $(a_1 + a_2, b)$ -Motzkin path P where each \mathbf{D} step is assigned with weight $a_1 + a_2$, by splitting into the summation of a_1 and a_2 we can assign each \mathbf{D} step by the weight either a_1 or a_2 . In this way we get $2^{\text{diag}(P)}$ $(\{a_1, a_2\}, b)$ -Motzkin paths.

Consider the set of all $(\{a_1, a_2\}, b)$ -Motzkin paths of order n . There are $\binom{n}{n-k}$ ways to choose $n - k$ \mathbf{D} steps among all the n steps, which are assigned with weight a_2 for each one. Each of the rest \mathbf{D} steps is assigned with weight a_1 . Each \mathbf{E}_2 step is assigned with weight b . Thus we can reduce a k -ordered (a_1, b) -Motzkin paths by removing all \mathbf{D} steps weighted by a_2 and get

$$\begin{aligned} M_n^{(a_1+a_2, b)} &= \sum_{P \in \text{Mot}_n(a_1+a_2, b)} \text{wt}(P) = \sum_{Q \in \text{Mot}_n(\{a_1, a_2\}, b)} \text{wt}(Q) \\ &= \sum_{k=0}^n \binom{n}{n-k} a_2^{n-k} \sum_{R \in \text{Mot}_k(a_1, b)} \text{wt}(R) \\ &= \sum_{k=0}^n \binom{n}{n-k} a_2^{n-k} M_k^{(a_1, b)}. \end{aligned}$$

□

Remark 2.2. There are some easy consequences of Lemma 2.1. Since $M_n^{(ax, bx^2)} = x^n M_n^{(a, b)}$ implies $M_n^{(a, b)} = (-1)^n M_n^{(-a, b)}$, by Lemma 2.1 we have

$$(2.2) \quad M_n^{(a, b)} = M_n^{(2a-a, b)} = \sum_{k=0}^n \binom{n}{k} M_k^{(-a, b)} (2a)^{n-k} = (-1)^n \sum_{k=0}^n \binom{n}{k} M_k^{(a, b)} (-2a)^{n-k}.$$

In particular, since $M_n = M_n^{(1, 1)}$, $C_{n+1} = M_n^{(2, 1)}$ and $S_{n+1} = 2M_n^{(3, 2)}$, by specializing a and b to be the corresponding values in the parenthesis we get

$$(2.3) \quad M_n = (-1)^n \sum_{k=0}^n (-2)^{n-k} \binom{n}{k} M_k,$$

$$(2.4) \quad C_{n+1} = (-1)^n \sum_{k=0}^n (-4)^{n-k} \binom{n}{k} C_{k+1},$$

$$(2.5) \quad S_{n+1} = (-1)^n \sum_{k=0}^n (-6)^{n-k} \binom{n}{k} S_{k+1}.$$

Now we are ready to prove Theorems 1.1 and 1.2. From (1.13) and (1.14) we know that

$$\sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^{2k} b^{n-1-k} = b^{-1} C_n^{(b, x^2)} = M_{n-1}^{(x^2+b, bx^2)}.$$

Using Lemma 2.1 and the identity $M_n^{(ax,bx^2)} = x^n M_n^{(a,b)}$ again, we get

$$\begin{aligned} M_{n-1}^{(x^2+b,bx^2)} &= \sum_{k=0}^{n-1} \binom{n-1}{k} M_k^{(ax,bx^2)} (x^2 - ax + b)^{n-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} M_k^{(a,b)} x^k (x^2 - ax + b)^{n-k}. \end{aligned}$$

Therefore Theorems 1.1 is obtained.

Similarly, from (1.12) and (1.16), we have

$$\begin{aligned} \sum_{k=0}^n \binom{n+k}{2k} C_k x^{2k} (b-x^2)^{n-k} &= S_n^{(b-x^2,x^2)} = b M_{n-1}^{(b+x^2,bx^2)} \\ &= b \sum_{k=0}^{n-1} \binom{n-1}{k} M_k^{(ax,bx^2)} (x^2 - ax + b)^{n-1-k} \\ &= b \sum_{k=0}^{n-1} \binom{n-1}{k} M_k^{(a,b)} x^k (x^2 - ax + b)^{n-1-k}. \end{aligned}$$

Therefore Theorem 1.2 holds. \square

3. MORE CONSEQUENCES OF THEOREMS 1.1-1.3

In this section, we shall list more identities involving C_n , M_n and S_n , which follows from Theorems 1.1-1.3.

Proposition 3.1. *For $a, b \in \mathbb{C}$,*

$$\begin{aligned} &\sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1}^{(a,b)} x^k (x-a)^{n-1-k} (x-b)^{n-1-k} \\ (3.1) \quad &= \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^{2k} a^{n-k} b^{n-1-k} \end{aligned}$$

$$(3.2) \quad = \frac{1}{b} \sum_{k=0}^n \binom{n+k}{2k} C_k x^{2k} (ab-x^2)^{n-k}.$$

Proof. By (1.14) and (1.7), we have

$$\begin{aligned} &\sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1}^{(a,b)} x^k (x-a)^{n-1-k} (x-b)^{n-1-k} \\ &\stackrel{(1.14)}{=} a \sum_{k=0}^{n-1} \binom{n-1}{k} M_k^{(a+b,ab)} x^k (x^2 - (a+b)x + ab)^{n-1-k} \\ &\stackrel{(1.7)}{=} \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^{2k} a^{n-k} b^{n-1-k}. \end{aligned}$$

Similarly, (3.2) can be obtained from (1.14) and (1.8). \square

In particular, substituting $a = b = 1$ in (3.2), we can get

$$(3.3) \quad \sum_{k=0}^n \binom{n+k}{2k} C_k x^{2k} (1-x^2)^{n-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1} x^k (x-1)^{2n-2-2k}.$$

By (1.15), which is $S_n^{(a,b)} = \frac{(a+b)}{b} C_n^{(b,a+b)}$, we may rewrite (3.1) and (3.2) respectively as follows:

$$(3.4) \quad \begin{aligned} & \sum_{k=0}^{n-1} \binom{n-1}{k} S_{k+1}^{(a,b)} x^k (x-b)^{n-1-k} (x-a-b)^{n-1-k} \\ &= \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^{2k} (a+b)^{n-k} b^{n-1-k}, \end{aligned}$$

$$(3.5) \quad \begin{aligned} & \sum_{k=0}^{n-1} \binom{n-1}{k} S_{k+1}^{(a,b)} x^k (x-b)^{n-1-k} (x-a-b)^{n-1-k} \\ &= \frac{1}{b} \sum_{k=0}^n \binom{n+k}{2k} C_k x^{2k} (ab+b^2-x^2)^{n-k}. \end{aligned}$$

Substituting $a = b = 1$ into (3.4) and (3.5), we have

$$(3.6) \quad \begin{aligned} & \sum_{k=1}^{n-1} \binom{n-1}{k} S_{k+1} x^k (x-1)^{n-1-k} (x-2)^{n-1-k} \\ &= \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^{2k} 2^{n-k} \end{aligned}$$

$$(3.7) \quad = \sum_{k=0}^n \binom{n+k}{2k} C_k x^{2k} (2-x^2)^{n-k}.$$

In particular, setting $x = \sqrt{2}$ in (3.6), we can get

$$(3.8) \quad C_n = \frac{(-\sqrt{2})^{n-1}}{2^n} \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} (\sqrt{2}-1)^{2(n-1-k)} S_{k+1}.$$

Similarly, applying (3.3) with $x = \frac{1}{2}\sqrt{2}$, we have

$$(3.9) \quad S_n = \sum_{k=0}^{n-1} 2^{\frac{k}{2}+1} \binom{n-1}{k} (\sqrt{2}-1)^{2(n-1-k)} C_{k+1}.$$

Of course, (3.9) also can be easily deduced from (3.8) via a binomial transform.

Proposition 3.2. For $\alpha, \beta \in \mathbb{C}$,

$$(3.10) \quad M_n^{(\alpha, \beta)} = \sum_{k=0}^{n+1} \left(\frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \right)^{n-2k} \cdot \frac{\beta^k}{n+1} \binom{n+1}{k} \binom{n+1}{k+1}$$

$$(3.11) \quad = \frac{(\alpha^2 - 4\beta)^{\frac{n+1}{2}}}{\beta} \sum_{k=0}^{n+1} \left(\frac{\alpha}{\sqrt{\alpha^2 - 4\beta}} - 1 \right)^{k+1} \cdot \frac{C_k}{2^{k+1}} \binom{n+k}{2k}.$$

Proof. Let $a = \frac{1}{2}(\alpha + \sqrt{\alpha^2 - 4\beta})$ and $b = \frac{1}{2}(\alpha - \sqrt{\alpha^2 - 4\beta})$. Evidently $a + b = \alpha$ and $ab = \beta$. So by (1.14)

$$\begin{aligned} M_n^{(\alpha, \beta)} &= \frac{C_{n+1}^{(a, b)}}{a} = a^n \sum_{k=0}^{n+1} \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k+1} \cdot \left(\frac{b}{a} \right)^k \\ &= \sum_{k=0}^{n+1} \left(\frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \right)^{n-2k} \cdot \frac{\beta^k}{n+1} \binom{n+1}{k} \binom{n+1}{k+1}. \end{aligned}$$

Similarly, letting $a = \sqrt{\alpha^2 - 4\beta}$ and $b = \frac{1}{2}(\alpha - \sqrt{\alpha^2 - 4\beta})$, we have $a + 2b = \alpha$ and $ab + b^2 = \beta$. Then

$$\begin{aligned} M_n^{(\alpha, \beta)} &= \frac{S_{n+1}^{(a, b)}}{a + b} = \frac{1}{a + b} \sum_{k=0}^{n+1} \binom{n+k}{2k} C_k \cdot a^{n-k} b^k \\ &= \frac{(\alpha^2 - 4\beta)^{\frac{n+1}{2}}}{\beta} \sum_{k=0}^{n+1} \left(\frac{\alpha}{\sqrt{\alpha^2 - 4\beta}} - 1 \right)^{k+1} \cdot \frac{C_k}{2^{k+1}} \binom{n+k}{2k}. \end{aligned}$$

□

In particular, substituting $\alpha = \beta = 1$ in (3.10) and (3.11), we obtain

$$(3.12) \quad M_n = \sum_{k=0}^{n+1} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \mathbf{i} \right)^{n-2k} \cdot \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k+1}$$

$$(3.13) \quad = \left(\frac{1}{2} - \frac{\sqrt{3}}{2} \mathbf{i} \right)^{n+2} \sum_{k=0}^{n+1} \left(-\frac{3}{2} + \frac{\sqrt{3}}{2} \mathbf{i} \right)^{n+1-k} \binom{n+1+k}{2k} C_k,$$

where $\mathbf{i} = \sqrt{-1}$.

Finally, let us see some applications of (2.1).

Proposition 3.3.

$$(3.14) \quad M_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} C_{k+1}$$

$$(3.15) \quad = \sum_{k=0}^n (-1)^k 3^{n-k} \binom{n}{k} C_{k+1}$$

$$(3.16) \quad = \frac{1}{(\sqrt{2})^{n+2}} \sum_{k=0}^n (\sqrt{2} - 3)^{n-k} \binom{n}{k} S_{k+1}$$

$$(3.17) \quad = \frac{1}{(\sqrt{2})^{n+2}} \sum_{k=0}^n (-1)^k (3 + \sqrt{2})^{n-k} \binom{n}{k} S_{k+1}.$$

Proof. By (2.1), we have

$$M_n^{(1,1)} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} M_k^{(2,1)} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} C_{k+1}.$$

This concludes (3.14). Similarly, (3.15) follows from that

$$\begin{aligned} M_n &= (-1)^n M_n^{(-1,1)} = (-1)^n M_n^{(2-3,1)} \\ &= (-1)^n \sum_{k=0}^n \binom{n}{k} M_k^{(2,1)} (-3)^{n-k} = \sum_{k=0}^n (-1)^k 3^{n-k} \binom{n}{k} C_{k+1}. \end{aligned}$$

On the other hand, since $M_k^{(3,2)} = \frac{1}{2} S_{k+1}$,

$$\begin{aligned} M_n &= \frac{1}{(\sqrt{2})^n} M_n^{(\sqrt{2},2)} = \frac{1}{(\sqrt{2})^n} M_n^{(3+\sqrt{2}-3,2)} \\ &= \frac{1}{(\sqrt{2})^n} \sum_{k=0}^n \binom{n}{k} M_k^{(3,2)} (\sqrt{2} - 3)^{n-k} \\ &= \frac{1}{(\sqrt{2})^{n+2}} \sum_{k=0}^n (\sqrt{2} - 3)^{n-k} \binom{n}{k} S_{k+1}. \end{aligned}$$

Also, we have

$$\begin{aligned} M_n &= \frac{M_n^{(-\sqrt{2},2)}}{(-\sqrt{2})^n} = \frac{M_n^{(3-\sqrt{2}-3,2)}}{(-\sqrt{2})^n} \\ &= \frac{1}{(\sqrt{2})^{n+2}} \sum_{k=0}^n (-1)^k \binom{n}{k} (\sqrt{2} + 3)^{n-k} S_{k+1}. \end{aligned}$$

Therefore (3.16) and (3.17) hold. \square

Using the fact $S_n = 2M_{n-1}^{(3,2)}$ again, together with the identities $M_n^{(\sqrt{2},2)} = 2^{\frac{n}{2}} M_n^{(1,1)}$ and $M_n^{(-\sqrt{2},2)} = (-1)^n M_n^{(\sqrt{2},2)}$, we can also get

$$(3.18) \quad \begin{aligned} S_n &= 2M_{n-1}^{(3,2)} = 2M_{n-1}^{(3-\sqrt{2}+\sqrt{2},2)} \\ &= \sum_{k=0}^{n-1} 2^{\frac{k}{2}+1} (3-\sqrt{2})^{n-1-k} \binom{n-1}{k} M_k \end{aligned}$$

$$(3.19) \quad \begin{aligned} S_n &= 2M_{n-1}^{(3,2)} = 2M_{n-1}^{(3+\sqrt{2}-\sqrt{2},2)} \\ &= \sum_{k=0}^{n-1} (-1)^k 2^{\frac{k}{2}+1} (3+\sqrt{2})^{n-1-k} \binom{n-1}{k} M_k. \end{aligned}$$

Remark 3.4. There are two arithmetic consequences of Theorem 3.3. In view of (3.15), we have

$$(3.20) \quad M_n \equiv (-1)^n \sum_{k=0}^{m-1} (-3)^k \binom{n}{k} C_{n-k+1} \pmod{3^m}$$

for each $m \geq 1$. In particular,

$$(3.21) \quad M_n \equiv (-1)^n C_{n+1} \pmod{3}.$$

Moreover, since $7 = (3 + \sqrt{2})(3 - \sqrt{2})$, by (3.16), we can get

$$(3.22) \quad M_n \equiv 2^{n+2} S_{n+1} \pmod{7}.$$

4. A BIJECTION ON THE SET OF CATALAN PATHS

Assume that $P = p_1 p_2 \dots p_{2n}$ is a Catalan path of order n and $a, b \in \mathbb{C}$. For a Catalan path $P = p_1 p_2 \dots p_{2n}$, a step p_i is said to be in an even (resp. odd) position if i is even (resp. odd). We shall assign the steps of P in even positions with some weights. For $1 \leq i \leq n$, assign the step p_{2i} with weight a or b according to whether $p_{2i} = \mathbf{E}$ or \mathbf{N} . The weighted Catalan path obtained in this way is called the even-north type (a, b) -Catalan path. Let $\text{CAT}_n(a, b)$ denote the set of all even-north type (a, b) -Catalan paths of order n and $\mathcal{C}_n^{(a,b)}$ the weight of $\text{CAT}_n(a, b)$. Let $\text{enor}(P)$ denote the number of \mathbf{N} steps in even positions on a Catalan path P . It's been proved that the statistic enor is distributed by the Narayana numbers [4, 5] and hence we have

$$\begin{aligned} \mathcal{C}_n^{(a,b)} &:= \sum_{P \in \text{CAT}_n(a,b)} \text{wt}(P) \\ &= \sum_{P \in \text{CAT}_n(a,b)} a^{n-\text{enor}(P)} b^{\text{enor}(P)} \\ &= \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} a^{n-k} b^k. \end{aligned}$$

We can easily see from the algebraic expression that the following theorem holds.

Theorem 4.1.

$$(4.1) \quad \mathcal{C}_n^{(a,b)} = C_n^{(a,b)}.$$

However, we are more interested in finding a purely combinatorial proof. That is, we shall construct a bijection between two different types of (a, b) -Catalan paths since we believe it will be also meaningful on its own sense. To give a bijective proof of (4.1), we first show the following lemma.

Lemma 4.2. *For any $0 \leq k \leq n-1$, there is a bijection between the set*

$$A_k := \{P \in \text{Cat}_n(a, b) : \text{valley}(P) = k\}$$

and the set

$$B_k := \{P \in \text{CAT}_n(a, b) : \text{enor}(P) = k\}.$$

Proof. Define $\phi : A_k \rightarrow B_k$ as follows:

Step 1. Notice that each path $P \in \text{Cat}(n)$ starting from $(0, 0)$ and ending at (n, n) always contains n **N** and **E** steps respectively. We can assume that $p_{s_1}, p_{s_2}, \dots, p_{s_n}$ are the successive **N** steps of P from bottom to top, and $p_{t_1}, p_{t_2}, \dots, p_{t_n}$ the successive **E** steps of P successively from left to right.

For example, Figure 1 gives a Catalan path $P = \mathbf{NENNEEENNEE}$ of order 6, whose **N** steps are $p_1, p_3, p_4, p_6, p_9, p_{10}$ and **E** steps are $p_2, p_5, p_7, p_8, p_{11}, p_{12}$. So we have

$$\begin{aligned} P &= p_1 p_2 p_3 p_4 \dots p_{11} p_{12} \\ &= \mathbf{NENNEEENNEE} \\ &= p_{s_1} p_{t_1} p_{s_2} p_{s_3} p_{t_2} p_{s_4} p_{t_3} p_{t_4} p_{s_5} p_{s_6} p_{t_5} p_{t_6}. \end{aligned}$$

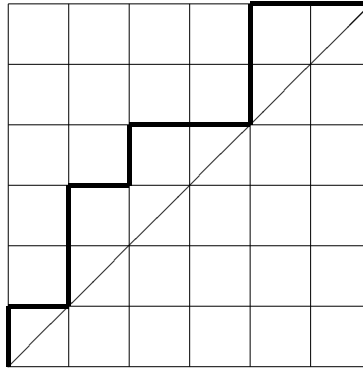


FIGURE 1. a Catalan path of order 6 with 3 valleys.

Step 2. Clearly each valley in P must be formed by some p_{t_i} and p_{s_j} with $s_j = t_i + 1$. Change p_{t_i} by an **N** step and change p_{s_j} by an **E** step, i.e., replace the valley $p_{t_i} p_{s_j}$ by a peak. Do such a transformation for all valleys in P , and denote the resultant path by $\hat{P} = \hat{p}_1 \hat{p}_2 \dots \hat{p}_{2n}$.

For example, for the Catalan path P in Figure 1, there are three valleys p_2p_3 , p_5p_6 , p_8p_9 in P . So

$$\hat{P} = \hat{p}_1\hat{p}_2\hat{p}_3\hat{p}_4 \cdots \hat{p}_{11}\hat{p}_{12} = \mathbf{N}\mathbf{N}\mathbf{E}\mathbf{N}\mathbf{N}\mathbf{E}\mathbf{E}\mathbf{N}\mathbf{E}\mathbf{N}\mathbf{E}\mathbf{E}.$$

Step 3. Let

$$\phi(P) = \hat{p}_{s_1}\hat{p}_{t_1}\hat{p}_{s_2}\hat{p}_{t_2} \cdots \hat{p}_{s_n}\hat{p}_{t_n}.$$

That is, $\phi(P)$ is obtained by alternately placing $\hat{p}_{s_1}, \dots, \hat{p}_{s_n}$ and $\hat{p}_{t_1}, \dots, \hat{p}_{t_n}$.

For example in Figure 1, we have

$$\phi(P) = \hat{p}_{s_1}\hat{p}_{t_1}\hat{p}_{s_2}\hat{p}_{t_2} \cdots \hat{p}_{s_6}\hat{p}_{t_6} = \mathbf{N}\mathbf{N}\mathbf{E}\mathbf{N}\mathbf{N}\mathbf{E}\mathbf{E}\mathbf{N}\mathbf{E}\mathbf{N}\mathbf{E}.$$

Clearly $\phi(P)$ has three \mathbf{N} steps in even positions, which are the 2^{nd} , 4^{th} and 8^{th} steps respectively as shown in Figure 2.

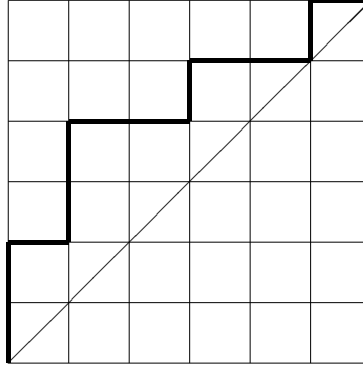


FIGURE 2. a Catalan path of order 6 with 3 EN-steps.

We may write $\phi(P) = q_1q_2 \cdots q_{2n}$. It is not difficult to see that $\text{wt}(\phi(P)) = \text{wt}(P)$ and $\text{enor}(\phi(P)) = \text{valley}(P)$. However, we still need to show that $\phi(P)$ is a Catalan path. That is equivalent to prove that for any $1 \leq h \leq 2n$, the following inequality holds:

$$(4.2) \quad |\{1 \leq i \leq h : q_i = \mathbf{N}\}| \geq |\{1 \leq j \leq h : q_j = \mathbf{E}\}|.$$

Clearly we always have $q_{2i-1} = \mathbf{N}$ and $q_{2i} = \mathbf{E}$, unless $q_{2i-1} = \mathbf{E}$ or $q_{2i} = \mathbf{N}$. Since $q_{2j-1} = \hat{p}_{s_j}$, if $q_{2j-1} = \mathbf{E}$, then $p_{s_j-1}p_{s_j}$ must be a valley in P , i.e., $t_i = s_j - 1$ for some $1 \leq i \leq n$. Since P is a Catalan path, we must have $j \geq i + 1$. Thus for any $1 \leq j \leq n$ with $q_{2j-1} = \mathbf{E}$, there exists a unique i such that $1 \leq i \leq j - 1$ and $q_{2i} = \hat{p}_{t_i} = \mathbf{N}$. Hence the number of \mathbf{N} steps is always not less than the number of \mathbf{E} steps in $\phi(P)$. Therefore (4.2) is valid.

Conversely, in order to show that ϕ is a bijection, we need to prove that for each $Q \in B_k$, there exists a unique $P \in A_k$ such that $Q = \phi(P)$. To do that, we orderly pair each evenly positioned \mathbf{N} step and each oddly positioned \mathbf{E} step of Q together to get all valley points in P . A valley point is the vertex common to both steps of the valley. Explicitly, assume that $Q = q_1q_2 \cdots q_{2n} \in B_k$ where $q_{2s_1-1}, \dots, q_{2s_k-1} = \mathbf{E}$ and $q_{2t_1}, \dots, q_{2t_k} = \mathbf{N}$. Then the valley points are formed by $(t_1, s_1 - 1), (t_2, s_2 - 1), \dots, (t_k, s_k - 1)$. Since Q is a Catalan path, we must have $t_i < s_i$ for each $1 \leq i \leq k$. We construct $P = p_1p_2 \cdots p_{2n}$ as follows:

First, set $p_1 = \cdots = p_{s_1-1} = \mathbf{N}$, $p_{s_1} = \cdots = p_{s_1+t_1-1} = \mathbf{E}$ and $p_{s_1+t_1} = \mathbf{N}$. The first valley is formed by steps $p_{s_1+t_1-1}p_{s_1+t_1}$. Next, set $p_{s_1+t_1+1}, \dots, p_{s_2+t_1-1} = \mathbf{N}$, $p_{s_2+t_1}, \dots, p_{s_2+t_2-1} = \mathbf{E}$ and $p_{s_2+t_2} = \mathbf{N}$. The second valley is formed by steps $p_{s_2+t_2-1}p_{s_2+t_2}$. Keep this process, until we get the last valley formed by $p_{t_k+s_k-1}p_{t_k+s_k}$. Finally, set $p_{s_k+t_k+1}, \dots, p_{n+t_k} = \mathbf{N}$ and $p_{n+t_k+1}, \dots, p_{2n} = \mathbf{E}$.

Note that for each $1 \leq i \leq k$, we have

$$|\{1 \leq j < s_i + t_i : p_j = \mathbf{N}\}| - |\{1 \leq j < s_i + t_i : p_j = \mathbf{E}\}| = (s_i - 1) - t_i \geq 0.$$

So P is also a Catalan path. It is not difficult to check that $\phi(P) = Q$. Hence ϕ is surely a bijection.

For example in Figure 2, since $q_3 = q_7 = q_9 = \mathbf{E}$ and $q_2 = q_4 = q_8 = \mathbf{N}$, the valley points are $(1, 1), (2, 3), (4, 4)$. Therefore, we have $p_1 = \mathbf{N}$, $p_2 = \mathbf{E}$, $p_3 = p_4 = \mathbf{N}$, $p_5 = p_6 = \mathbf{N}$, $p_7 = p_8 = \mathbf{E}$, $p_9 = p_{10} = \mathbf{N}$, $p_{11} = p_{12} = \mathbf{E}$, which form exactly the path P in Figure 1. \square

Notice that from the proof of Lemma 4.2, we have $\text{wt}(\phi(P)) = \text{wt}(P)$ for each $P \in A_k$. So (4.1) immediately follows from Lemma 4.2.

5. PROOFS OF (1.14) AND (1.15)

Proof of (1.14). By Theorem 4.1, to show (1.14) is equivalent to prove the following identity

$$(5.1) \quad \mathcal{C}_n^{(a,b)} = aM_{n-1}^{(a+b,ab)}.$$

Let $\text{Mot}_n(\{a, b\}, a, b)$ denote the set of all weighted n -ordered Motzkin paths, with each \mathbf{E}_2 step assigned with weight a , each \mathbf{N}_2 step assigned with weight b , and each \mathbf{D} step assigned with weight either a or b . For a path $P \in \text{Mot}_{n-1}(\{a, b\}, a, b)$, we can remove the weights of \mathbf{N}_2 step and reassign each \mathbf{E}_2 step with weight ab . Then we get an $(\{a, b\}, ab)$ -Motzkin path of order $n - 1$. Evidently the two paths have the same weight, since the number of \mathbf{N}_2 steps and \mathbf{E}_2 steps in a Motzkin path are always equal. So we have

$$\sum_{P \in \text{Mot}_{n-1}(\{a,b\},a,b)} \text{wt}(P) = \sum_{P \in \text{Mot}_{n-1}(\{a,b\},ab)} \text{wt}(P).$$

On the other hand, from the discussion in the proof of Lemma 2.1, we have

$$M_{n-1}^{(a+b,ab)} = \sum_{P \in \text{Mot}_{n-1}(a+b,ab)} \text{wt}(P) = \sum_{P \in \text{Mot}_{n-1}(\{a,b\},ab)} \text{wt}(P).$$

It suffices to give a bijection $\psi : \text{CAT}_n(a, b) \rightarrow \text{Mot}_{n-1}(\{a, b\}, a, b)$. For a path $P = p_1 p_2 \dots p_{2n} \in \text{CAT}_n(a, b)$, we know that $p_1 = \mathbf{N}$ and $p_{2n} = \mathbf{E}$ with weight a . Let $\psi(P) = q_1 q_2 \dots q_{n-1} \in \text{Mot}_n(\{a, b\}, a, b)$ be given as follows: for $1 \leq i \leq n - 1$,

$p_{2i} p_{2i+1} = \mathbf{NN}$ with weight b if and only if $q_i = \mathbf{N}_2$ with weight b ;

$p_{2i} p_{2i+1} = \mathbf{EE}$ with weight a if and only if $q_i = \mathbf{E}_2$ with weight a ;

$p_{2i} p_{2i+1} = \mathbf{NE}$ with weight b if and only if $q_i = \mathbf{D}$ with weight b ;

$p_{2i} p_{2i+1} = \mathbf{EN}$ with weight a if and only if $q_i = \mathbf{D}$ with weight a .

Clearly ψ is a bijection. Since the last step p_{2n} is always **E** with weight a , we have $\text{wt}(P) = a \cdot \text{wt}(\psi(P))$. Therefore, we have

$$\mathcal{C}_n^{(a,b)} = \sum_{P \in \text{CAT}_n(a,b)} \text{wt}(P) = a \left(\sum_{\psi(P) \in \text{Mot}_{n-1}(\{a,b\},a,b)} \text{wt}(\psi(P)) \right) = a M_{n-1}^{(a+b,ab)}.$$

□

Proof of (1.15). A peak in a Catalan path is defined to be an **N** step followed immediately by an **E** step. Assume that $P = p_1 p_2 \cdots p_{2n}$ is a Catalan path of order n . For $1 \leq i \leq n-1$, assign each **E** step p_i with weight b provided that $p_i p_{i+1} = \mathbf{NE}$ which forms a peak in P . Also, assign each of the rest **E** step of P with weight a . Thus we get a *peak type (a,b) -Catalan path*. Let $\overline{\text{Cat}}_n(a,b)$ be the set of all peak type (a,b) -Catalan paths of order n , and $\text{peak}(P)$ denote the number of all peaks in P . Clearly $\text{peak}(P) = \text{valley}(P) + 1$. Hence

$$\begin{aligned} \sum_{P \in \overline{\text{Cat}}_n(a,b)} \text{wt}(P) &= \sum_{P \in \overline{\text{Cat}}_n(a,b)} a^{n-\text{peak}(P)} b^{\text{peak}(P)} \\ &= \sum_{P \in \overline{\text{Cat}}_n(a,b)} a^{n-1-\text{valley}(P)} b^{\text{valley}(P)+1} \\ (5.2) \quad &= \frac{b}{a} \sum_{P \in \overline{\text{Cat}}_n(a,b)} a^{n-\text{valley}(P)} b^{\text{valley}(P)} = \frac{b}{a} \cdot C_n^{(a,b)}. \end{aligned}$$

For P an n -ordered Catalan path, if we assign each **E** step of a peak with weight either b_1 or b_2 , and each other **E** step of P with weight a , then the resultant weighted path is called a peak type $(a, \{b_1, b_2\})$ -Catalan paths of order n . Let $\overline{\text{Cat}}_n(b, \{a, b\})$ denote the set of all peak type $(b, \{a, b\})$ -Catalan paths of order n . For a peak type $(b, a+b)$ Catalan path P , by splitting the weight $a+b$ into the summation of a and b , we can get $2^{\text{peak}(P)}$ peak type $(b, \{a, b\})$ -Catalan paths. So

$$(5.3) \quad \sum_{P \in \overline{\text{Cat}}_n(b, a+b)} \text{wt}(P) = \sum_{P \in \overline{\text{Cat}}_n(b, \{a, b\})} \text{wt}(P).$$

In view of (5.2) and (5.3), to prove (1.15), we only need to find a bijection from $\text{Sch}_n(a,b)$ to $\overline{\text{Cat}}_n(b, \{a, b\})$. Such a bijection can be constructed in a natural way. For an (a,b) -Schröder path $P \in \text{Sch}_n(a,b)$, since each **D** step in P is assigned with weight a , we can then replace each **D** step by an adjacent **EN** pair and assign the **E** step of this peak with weight a . Doing this replacement for all **D** steps in the (a,b) -Schröder path P . We get a peak type $(b, \{a, b\})$ -Catalan path. Conversely, for a peak type $(b, \{a, b\})$ -Catalan path, we can replace each adjacent **EN** pair by a **D** step with weight a , provided that the **E** step is assigned with weight a . We also can get an (a,b) -Schröder path. Thus we surely obtain a bijection from $\text{Sch}_n(a,b)$ to $\overline{\text{Cat}}_n(b, \{a, b\})$. It follows that

$$\sum_{P \in \text{Sch}_n(a,b)} \text{wt}(P) = \sum_{P \in \overline{\text{Cat}}_n(b, \{a, b\})} \text{wt}(P) = \sum_{P \in \overline{\text{Cat}}_n(b, a+b)} \text{wt}(P) = \frac{a+b}{b} \cdot C_n^{(b, a+b)},$$

and therefore (1.15) holds. □

Remark 5.1. It can be seen that (1.16) easily follows from (1.14) and (1.15). In fact, using the same way in [10], we can also give a direct bijection proof of (1.16). However, since the proof is a little bit tedious and contains almost no new ideas, we omit it in this paper.

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